Worcester County Mathematics League

Varsity Meet 1
October 5, 2016

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016

Round 1: Arithmetic

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $a = \frac{2}{3}$, evaluate $a \div \left(1 - \frac{1}{a}\right)$.

2. If $x = 72$ and $y = 47$, evaluate $4x^2 - 9y^2$.

3. Let $b = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \ldots}}}}}}$. Evaluate $\frac{1}{b}$.

ANSWERS

(1 pt.) 1. ____________________________

(2 pts.) 2. ____________________________

(3 pts.) 3. ____________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016
Round 2: Algebra 1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. A piggy bank contains \$10.55 in dimes and quarters. If the number of quarters is 1 less than than twice the number of dimes, how many dimes are in the bank?

2. Compute the y-intercept of a line which has an x-intercept of -3 and is parallel to the line $2x-4y=8$.

3. When each of three numbers is added to the average of the other two numbers the respective sums are 26, 30, and 32. What is the largest of the three numbers?

ANSWERS

(1 pt.) 1. ________________________ dimes

(2 pts.) 2. ________________________

(3 pts.) 3. ________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016
Round 3: Set Theory

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. The town of Ware has exactly 5000 families. Every family owns either 0, 1, or 2 pets. Every pet in Ware belongs to a family. Most families own 1 pet and exactly half of the remaining families own 2 pets. How many pets in total live in Ware?

2. Let set A be all integers divisible by 7 between 199 and 501. Let set B be all integers divisible by 5 between 199 and 501. How many elements belong to the set $A \cup B$?

3. 200 students took a 3 question test. 50 students answered all three questions. 83 answered question 1, 78 answered question 2, and 88 answered question 3. 13 students answered only question 1, 13 students answered only question 2, and 13 students answered only question 3. How many students answered no questions?

ANSWERS

(1 pt.) 1. ________________ pets

(2 pts.) 2. ________________ elements

(3 pts.) 3. ________________ students

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016
Round 4: Measurement

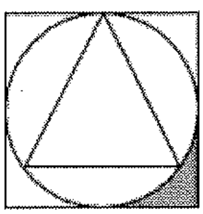
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If the area of a circle is numerically equal to 7 times its circumference, what is the diameter of the circle?

2. The length of a room exceeds its width by 5 feet and its height is 3 feet less than its width. The sum of the areas of the four walls exceeds the sum of the areas of the floor and ceiling by 240 square feet. What is the height of the room?

3. A square circumscribes a circle which circumscribes an equilateral triangle, as shown in the diagram. If the triangle's area is $16\sqrt{3}$ square inches, find the area of the shaded region. Express your answer in terms of π.



ANSWERS

(1 pt.) 1. ____________________________ units

(2 pts.) 2. ____________________________ feet

(3 pts.) 3. ____________________________ square inches

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016
Round 5: Polynomial Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Suppose that -8 is one root of $5x^2 + 34x = k$. Determine the value of k.

2. Let $f(x)$ be the polynomial function of lowest degree which passes through the following points: (1, 0), (2, 4), (3, 0), (5, 0). Evaluate $f(0)$.

3. If a and b are the roots of the function $f(x) = x^2 - px - q$, there is a quadratic function $g(x)$ which has roots $a + \frac{1}{b}$ and $b + \frac{1}{a}$. If $g(x) = Dx^2 + Ex + F$, express E in terms of p and q.

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. $E =$ ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016

Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)

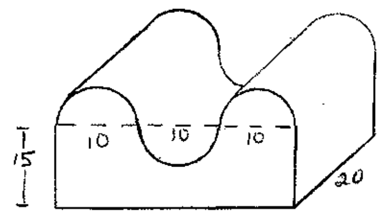
APPROVED CALCULATORS ALLOWED

1. If $a \circ b = 1 - \frac{a}{b}$ and $a \star b = \frac{a}{a-b}$, then evaluate: $\dfrac{2\circ[3\star(4\circ7)]}{2\star[3\circ(4\star7)]}$

2. Let x be a two digit base 10 integer. If 3 times the sum of its digits is added to x, the result is the original integer with its digits reversed. Find all possible values of x.

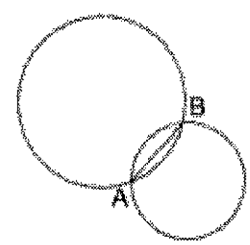
3. Let A be the set of positive integers. If $S = \{x \in A : |2x-7| \leq 5\}$ and $T = \{x \in A : |3x-14| < 9\}$, how many elements are in $S \cup T$?

4. Find the surface area of the figure to the right and express your answer in terms of π. Note that the curve along the top edge forms three semicircles each with diameter 10.



5. Find all real and complex solutions to $x^3 - x^2 + 2 = 0$.

6. Two circles overlap as shown. The radius of the larger circle is 2 units and the radius of the smaller circle is $\sqrt{2}$ units. If the length of AB is 2 units, what is the area of the region where the two circles overlap?



7. Factor completely: $A^2 + B^2 + 3A + 3B + 2AB - 4$

8. Laura can build a standard brick wall in exactly 9 hours by herself. Crystal can build the same size wall in 10 hours by herself. If they work together, their combined rate of labor decreases by 10 bricks laid per hour and they finish the wall in 5 hours. How many bricks are contained in one standard brick wall?

9. Joe needs to cut a board which is 300 cm long into three different size pieces. The ratio of the length of longest piece to the length of the shortest piece is 5 : 3 and the ratio of the length of the middle size piece to the length of the shortest piece is 3 : 2. Find the length of the shortest piece.

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016
Team Round Answer Sheet

1. ____________________________

2. ____________________________

3. ____________________________ elements

4. ____________________________ square units

5. $x =$ ____________________________

6. ____________________________ square units

7. ____________________________

8. ____________________________ bricks

9. ____________________________ cm

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016 ANSWER KEY

Round 1:

1. $-\frac{4}{3}$ or $-1\frac{1}{3}$ or $-1.\overline{3}$ (Bancroft)
2. 855 (Millbury)
3. $\frac{\sqrt{37}-4}{3}$ or $0.\overline{3}(\sqrt{37}-4)$ (Worcester Academy)

Round 2:

1. 18 (Bartlett)
2. $y=$ $\frac{3}{2}$ or 1 ½ or 1.5 (North)
3. 20 (Mass Academy)

Round 3:

1. 5000 (St. John's)
2. 95 (Tantasqua)
3. 81 (Bromfield)

Round 4:

1. 28 (St. John's)
2. 12 (Holy Name)
3. $\frac{64}{3}-\frac{16}{3}\pi$ or $21\frac{1}{3}-5\frac{1}{3}\pi$ or $21.\overline{3}-5.\overline{3}\pi$ (Algonquin)

Round 5:

1. 48 (Auburn)
2. -20 (Mass Academy)
3. $-\left(p+\frac{p}{q}\right)$ or $-p-\frac{p}{q}$ or $-\frac{pq+p}{q}$ or $\frac{-pq-p}{q}$ or $-\frac{p(q+1)}{q}$ (Doherty)

TEAM Round

1. $\frac{25}{56}$ or 0.446 (Doherty)
2. 12, 24, 36, and 48 (ARHS)
3. 7 (Bancroft)
4. $325\pi + 2100$ (Notre Dame Academy)
5. $-1, 1+i, 1-i$ (Norton)
6. $\frac{7}{6}\pi-1-\sqrt{3}$ or 0.933 (Hopkinton)
7. (A + B + 4)(A + B -1) (Worcester Academy)
8. 900 (Bromfield)
9. 72 (West Boylston)

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 - October 5, 2016 - SOLUTIONS

Round 1: Arithmetic

1. If $a = \frac{2}{3}$, evaluate $a \div \left(1 - \frac{1}{a}\right)$.

Solution: Substitute the value of a into the expression to get

$$\frac{2}{3} \div \left(1 - \frac{1}{\frac{2}{3}}\right)$$
$$\frac{2}{3} \div \left(1 - \frac{3}{2}\right)$$
$$\frac{2}{3} \div \left(-\frac{1}{2}\right)$$
$$\frac{2}{3} \times \left(-\frac{2}{1}\right) = \frac{-4}{3}$$

2. If $x = 72$ and $y = 47$, evaluate $4x^2 - 9y^2$.

Solution 1 (Straight Computation): Substitute the values of x and y into the expression and compute: $4(72)^2 - 9(47)^2$

$$4(5184) - 9(2209)$$
$$20736 - 19881 = 855$$

Solution 2 (Difference of Squares): First factor the expression and then evaluate:

$$4x^2 - 9y^2$$
$$(2x - 3y)(2x + 3y)$$
$$[2(72) - 3(47)][2(72) + 3(47)]$$
$$(144 - 141)(144 + 141)$$
$$3 \times 285 = 855$$

1. Let $b = 1 + \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\ldots}}}}}}$. Evaluate $\frac{1}{b}$.

Solution: Let x be equal to $\frac{1}{b}$. Notice that we can write

$$x = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+x}}}$$

$$\frac{1}{x} = 1 + \cfrac{1}{2+\cfrac{1}{3+x}}$$

Subtract 1 from both sides of the equation to get

$$\frac{1-x}{x} = \cfrac{1}{2+\cfrac{1}{3+x}}$$

$$\frac{1-x}{x} = \cfrac{1}{\frac{7+2x}{3+x}}$$

$$\frac{1-x}{x} = \frac{3+x}{7+2x}$$

$$(1-x)(7+2x) = x(3+x)$$

$$7 - 5x - 2x^2 = 3x + x^2$$

$$0 = 3x^2 + 8x - 7$$

Using the quadratic equation, we see that

$$x = \frac{-8 \pm \sqrt{64-4(3)(-7)}}{2(3)} = \frac{-8 \pm \sqrt{148}}{6}$$

Since the original expression is clearly non-negative, we have that

$$x = \frac{-8+\sqrt{148}}{6} = \frac{2\sqrt{37}-8}{6} = \frac{\sqrt{37}-4}{3}$$

Round 2: Algebra 1

1. A piggy bank contains \$10.55 in dimes and quarters. If the number of quarters is 1 less than than twice the number of dimes, how many dimes are in the bank?

Solution 1 (Immediate Substitution): Let x denote the number of dimes in the piggy bank. This means that the number of quarters is given by $2x - 1$. To determine the number of dimes we simply solve

$$0.10x + 0.25(2x - 1) = 10.55$$

$$0.10x + 0.5x - 0.25 = 10.55$$

$$0.60x = 10.80$$

$$60x = 1080$$

$$x = 18$$

Solution 2 (System of Equations): Let Q be the number of quarters in the man's pocket and D be the number of dimes. Then we have the following system of equations:

$$25Q + 10D = 1055$$
$$Q = 2D - 1$$

Substitute the second equation into the first to get that

$$25(2D-1) + 10D = 1055$$
$$50D - 25 + 10D = 1055$$
$$60D = 1080$$
$$D = 18.$$

2. Compute the y-intercept of a line which has an x-intercept of -3 and is parallel to the line $2x - 4y = 8$.

Solution 1: Begin by putting the given line into mx + b form:

$$2x - 4y = 8$$
$$-4y = -2x + 8$$
$$y = \tfrac{1}{2}x - 2$$

Now we can see that the desired line has a slope of 0.5 and from the statement of the problem we know that it crosses the point (-3,0). Therefore to determine the y-intercept, which we will call B, we need only solve

$$0 = \tfrac{1}{2}(-3) + B$$
$$0 = \tfrac{-3}{2} + B$$
$$B = \tfrac{3}{2}$$

Solution 2 (Point Slope): Find the desired slope in the same way as in solution 1. Once we know that the desired slope is $\frac{1}{2}$, use the fact that the x-intercept is the point (-3,0) to construct the desired line in point slope form:

$$y - 0 = \tfrac{1}{2}(x - (-3))$$
$$y = \tfrac{1}{2}x + \tfrac{3}{2}.$$

Therefore, we have that the y-intercept is $\frac{3}{2}$.

3. When each of three numbers is added to the average of the other two numbers the respective sums are 26, 30, and 32. What is the largest of the three numbers?

Solution: Let $a, b,$ and c be the three numbers, where c is the largest and a is the smallest. We can write the following three expressions which show each of the three numbers added to the average of the other two.

(i) $a+\frac{b+c}{2}$

(ii) $b+\frac{a+c}{2}$

(iii) $c+\frac{a+b}{2}$

If we double each of the above expressions we get:

(i.a) $2a+b+c=(a+b+c)+a$

(ii.a) $a+2b+c=(a+b+c)+b$

(iii.a) $a+b+2c=(a+b+c)+c$

Since c is the largest of the numbers, we can therefore conclude that expression (iii) must have the largest value among expressions (i), (ii) and (iii). Similarly, since a is the smallest of the numbers, we can conclude that (i) must have the smallest value. Hence, we can now set up the following system of equations:

$$a+\frac{b+c}{2}=26 \rightarrow 2a+b+c=52 \qquad (1)$$

$$b+\frac{a+c}{2}=30 \rightarrow a+2b+c=60 \qquad (2)$$

$$c+\frac{a+b}{2}=32 \rightarrow a+b+2c=64 \qquad (3)$$

Subtracting equation (1) from equation (3) we get

$$c-a=12 \rightarrow a=c-12 \qquad (4)$$

Subtracting equation (1) from equation (2) we get

$$b-a=8 \rightarrow b=a+8 \qquad (5)$$

Combining equations (4) and (5) gives

$$b=(c-12)+8=c-4 \qquad (6)$$

Finally, we can substitute equations (4) and (6) into equation (3) to get

$$(c-12)+(c-4)+2c=64$$

$$4c-16=64$$

$$4c=80 \quad \rightarrow c=20$$

Round 3: Set Theory

1. The town of Ware has exactly 5000 families. Every family owns either 0, 1, or 2 pets. Every pet in Ware belongs to a family. Most families own 1 pet and exactly half of the remaining families own 2 pets. How many pets in total live in Ware?

Solution: Let x denote the total number of households which own either 2 pets or 0 pets. Notice that since there are equal numbers of households which own 2 pets and households which own 0 pets, we know the total number of pets living in either 0 or 2-pet homes is precisely $\frac{1}{2}x(2)+\frac{1}{2}x(0)=x$.

We know that there are $5000-x$ households which contain exactly one pet, so the number of pets living in 1-pet homes is precisely $5000-x$. Since every pet in Ware belongs to a home, the total number of pets is given by $x+(5000-x)=$ **5000 pets.**

2. Let set A be all integers divisible by 7 between 199 and 501. Let set B be all integers divisible by 5 between 199 and 501. How many elements belong to the set $A \cup B$?

Solution: Let $n(A)$ be the number of elements in set A. By the inclusion-exclusion principle, we know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

To determine $n(A)$, note that $196 \div 7 = 28$ and $497 \div 7 = 71$. This means that there are $71 - 28 = 43$ multiples of 7 between 199 and 501.

To determine $n(B)$, note that $195 \div 5 = 39$ and $500 \div 5 = 100$. This means that there are $100 - 39 = 61$ multiples of 5 between 199 and 501.

Finally, to determine $n(A \cap B)$, note that $5 \times 7 = 35$, $175 \div 35 = 5$ and $490 \div 35 = 14$. This means that there are $14 - 5 = 9$ multiples of 35 between 199 and 501. In other words, there are 9 numbers between 199 and 501 divisible by both 5 and 7.

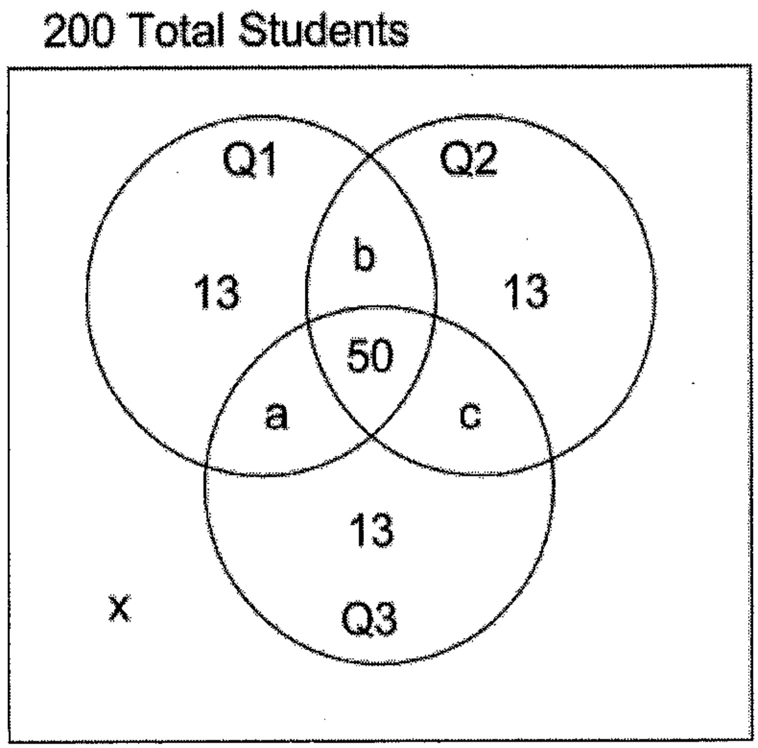
To conclude, we have that $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 43 + 61 - 9 = 95$.

3. 200 students took a 3 question test. 50 students answered all three questions. 83 answered question 1, 78 answered question 2, and 88 answered question 3. 13 students answered only question 1, 13 students answered only question 2, and 13 students answered only question 3. How many students answered no questions?

Solution 1: Begin by drawing a Venn diagram of the known information.

Let a be the number of students that answered only questions 1 and 3. Let b be the number that only answered questions 1 and 2. Let c be the number that only answered questions 2 and 3.

Finally let x be the number of students which answered exactly 0 questions.



Since 83 answered question 1, we know that $a+b=20$. (1)
Since 78 answered question 2, we know that $b+c=15$. (2)
Since 88 answered question 3, we know that $a+c=25$. (3)

Equation (2) gives us that $b=15-c$. (4)
Equation (3) gives us that $a=25-c$. (5)

Substituting equations (4) and (5) into equation (1) gives us that

$$(25-c)+(15-c)=20$$
$$40-2c=20$$
$$20=2c \rightarrow c=10.$$

From equations (4) and (5) this immediately gives that $a=15$ and $b=5$. Therefore, we have that the number of students who answered 0 questions is

$$x=200-50-20-15-10-13-13-13=81.$$

Solution 2: Use the same notation as in Solution 1 and derive the same initial three equations:

$$a+b=20 \qquad (1)$$

$$b+c=15 \qquad (2)$$

$$a+c=25 \qquad (3)$$

Adding equations (1), (2) and (3) gives us that $2a+2b+2c=60 \rightarrow a+b+c=30$. Now we can solve for x:

$$200=x+(a+b+c)+50+39$$

$$200=x+30+50+39$$

$$x=81.$$

Round 4: Measurement

1. If the area of a circle is numerically equal to 7 times its circumference, what is the diameter of the circle?

Solution: If r is the radius of the circle, we have that

$$\pi r^2 = 7\times 2\pi r$$

$$\pi r = 14\pi$$

$$r=14$$

This means the diameter of the circle is 28 units.

2. The length of a room exceeds its width by 5 feet and its height is 3 feet less than its width. The sum of the areas of the four walls exceeds the sum of the areas of the floor and ceiling by 240 square feet. What is the height of the room?

Solution: Let x denote the width of the room. We therefore have that the length of the room is $x+5$ and the height of the room is $x-3$. The rest of the statement of the problem tells us that

$$2(x-3)(x+5)+2x(x-3)=2x(x+5)+240$$

$$2(x^2+2x-15)+2x^2-6x=2x^2+10x+240$$

$$2x^2+4x-30-6x=10x+240$$

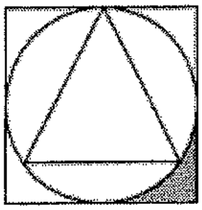
$$2x^2-12x-270=0$$

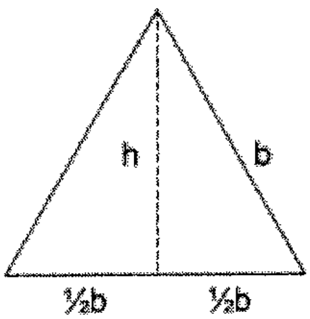
$$x^2-6x-135=0$$

$$(x+9)(x-15)=0$$

Since x must be positive, we have that x, the room's length, is 15 ft. This means the room's height is 12 ft.

3. A square circumscribes a circle which circumscribes an equilateral triangle, as shown in the diagram. If the triangle's area is $16\sqrt{3}$ square inches, find the area of the shaded region. Express your answer in terms of π.





Solution 1: Let b be the length of the edge of the equilateral triangle and h be its height, as we can see in the diagram to the left. Since this is a $60^\circ-60^\circ-60^\circ$ triangle, if we split it along the dotted line, it forms two identical $30^\circ-60^\circ-90^\circ$ triangles with base $\frac{1}{2}b$, height h, and hypotenuse b.

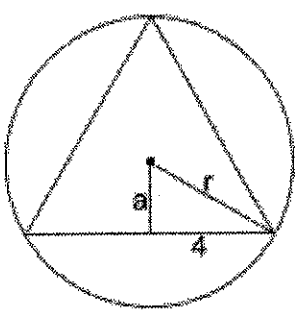
By the Pythagorean Theorem, we have that $h=\sqrt{b^2-\frac{1}{4}b^2}=\frac{\sqrt{3}}{2}b$.

Now to determine b we use the formula for the area of a triangle

$$A=\tfrac{1}{2}bh$$
$$A=\tfrac{1}{2}b\left(\tfrac{\sqrt{3}}{2}b\right)=\tfrac{\sqrt{3}}{4}b^2$$

We are given that the area of the equilateral triangle is $16\sqrt{3}$. Therefore

$$16\sqrt{3}=\tfrac{\sqrt{3}}{4}b^2$$
$$16\times 4=b^2$$
$$b^2=64\rightarrow b=8.$$



Now that we know b we need to compute r, the radius of the circle. Since the triangle is equilateral, we know that the small triangle drawn in the diagram to the left has exactly one sixth of the area of the total triangle. This allows us to set up the following equation:

$$\tfrac{1}{6}\times 16\sqrt{3}=\tfrac{1}{2}(a)(4)$$
$$16\sqrt{3}=12a$$
$$a=\tfrac{16}{12}\sqrt{3}=\tfrac{4}{3}\sqrt{3}.$$

Using the Pythagorean Theorem again we have that

$$r=\sqrt{4^2+\left(\tfrac{4}{3}\sqrt{3}\right)^2}=\sqrt{16+\tfrac{16}{3}}=\sqrt{\tfrac{64}{3}}=8\tfrac{\sqrt{3}}{3}$$

This means that side of the square is $16\frac{\sqrt{3}}{3}$ and has an area of $\frac{256}{3}$ sq inches. The area of the circle is $\frac{64}{3}\pi$ sq inches. Finally, to find the area of the shaded region, we need to compute one fourth of the the difference between the square's area and the circle's area.

$$\text{Desired Area}=\tfrac{1}{4}\left(\tfrac{256}{3}-\tfrac{64}{3}\pi\right)=\tfrac{64}{3}-\tfrac{16}{3}\pi \text{ square inches.}$$

Solution 2 (Equilateral Triangle Facts): Use the same notation as in Solution 1. It is a fact that the area of an equilateral triangle with side b is given by $\frac{\sqrt{3}}{4}b^2$. Therefore, we have that

$$16\sqrt{3} = \frac{\sqrt{3}}{4}b^2$$
$$64 = b^2 \rightarrow b = 8.$$

Now, since we know b and the area of the triangle, we can compute

$$\frac{1}{2}bh = 16\sqrt{3}$$
$$4h = 16\sqrt{3} \rightarrow h = 4\sqrt{3}.$$

Next, use the fact that two thirds of the height of an equilateral triangle inscribed in a circle is equal to the circle's radius:

$$r = \frac{2}{3}h \rightarrow r = \frac{8\sqrt{3}}{3}.$$

This means that side of the square is $16\frac{\sqrt{3}}{3}$ and has an area of $\frac{256}{3}$ sq inches. The area of the circle is $\frac{64}{3}\pi$ sq inches. Finally, to find the area of the shaded region, we need to compute one fourth of the the difference between the square's area and the circle's area.

$$\text{Desired Area} = \frac{1}{4}\left(\frac{256}{3} - \frac{64}{3}\pi\right) = \frac{64}{3} - \frac{16}{3}\pi \text{ square inches.}$$

Round 5: Polynomial Equations

1. Suppose that -8 is one root of $5x^2 + 34x = k$. Determine the value of k.

Solution: If -8 is a root of the given polynomial, then we know that

$$5(-8)^2 + 34(-8) - k = 0$$
$$5(64) - 272 - k = 0$$
$$320 - 272 = k$$
$$k = 48.$$

2. Let $f(x)$ be the polynomial function of lowest degree which passes through the following points: (1, 0), (2, 4), (3, 0), (5, 0). Evaluate $f(0)$.

Solution: We are given that the polynomial has at least three roots, so we know that the lowest possible degree it could have is three. One such 3rd degree polynomial that passes through the given roots is $f(x) = (x-1)(x-3)(x-5)$.

We need to check, however, that this possible polynomial passes through the point (2, 4).

$$f(2) = (2-1)(2-3)(2-5)$$
$$f(2) = (1)(-1)(-3) = 3.$$

However, we can simply scale this polynomial by multiplying it by a factor of $\frac{4}{3}$ so that it will now cross the point (2, 4). To conclude, we evaluate this polynomial at $x = 0$:

$$f(0) = \frac{4}{3}(0-1)(0-3)(0-5) = \frac{4}{3}(-15) = -20.$$

3. If a and b are the roots of the function $f(x)=x^2-px-q$, there is a quadratic function $g(x)$ which has roots $a+\frac{1}{b}$ and $b+\frac{1}{a}$. If $g(x)=Dx^2+Ex+F$, express E in terms of p and q.

Solution: If a and b are roots of $f(x)$, then we know we can write

$$f(x)=(x-a)(x-b)$$
$$f(x)=x^2-(a+b)x+ab.$$

This gives us that $p=a+b$ and $q=ab$.

Now we are interested in the quadratic function $g(x)$ which has roots $a+\frac{1}{b}$ and $b+\frac{1}{a}$. This function can be expressed as

$$g(x)=(x-(a+\tfrac{1}{b}))(x-(b+\tfrac{1}{a}))$$
$$g(x)=x^2-(b+\tfrac{1}{a})x-(a+\tfrac{1}{b})x+(ab+1+1+\tfrac{1}{ab})$$
$$g(x)=x^2-(a+b+\tfrac{1}{a}+\tfrac{1}{b})x+ab+\tfrac{1}{ab}+2$$
$$g(x)=x^2-(a+b+\tfrac{a+b}{ab})x+ab+\tfrac{1}{ab}+2$$
$$g(x)=x^2-(p+\tfrac{p}{q})x+q+\tfrac{1}{q}+2.$$

Therefore, $E=-\left(p+\frac{p}{q}\right)$

Team Round

1. If $a\circ b=1-\frac{a}{b}$ and $a\star b=\frac{a}{a-b}$, then evaluate: $\dfrac{2\circ[3\star(4\circ7)]}{2\star[3\circ(4\star7)]}$

Solution: Work from the inside out, starting with the numerator.

$$4\circ7=1-\tfrac{4}{7}=\tfrac{3}{7}$$
$$3\star\tfrac{3}{7}=\frac{3}{3-\frac{3}{7}}=\frac{3}{\frac{18}{7}}=\tfrac{7}{6}$$
$$2\circ\tfrac{7}{6}=1-\frac{2}{\frac{7}{6}}=1-\tfrac{12}{7}=\tfrac{-5}{7}$$

Now for the denominator we have

$$4\star7=\tfrac{4}{4-7}=\tfrac{-4}{3}$$
$$3\circ\tfrac{-4}{3}=1-\frac{3}{\frac{-4}{3}}=1+\tfrac{9}{4}=\tfrac{13}{4}$$
$$2\star\tfrac{13}{4}=\frac{2}{2-\frac{13}{4}}=\frac{2}{-\frac{5}{4}}=\tfrac{-8}{5}$$

To conclude we calculate: $\frac{-5}{7}\div\frac{-8}{5}=\frac{-5}{7}\times\frac{5}{-8}=\frac{25}{56}$

2. Let x be a two digit base 10 integer. If 3 times the sum of its digits is added to x, the result is the original integer with its digits reversed. Find all possible values of x.

Solution: Let a be the number's tens digit and b be the number's units digit. We then can write that $x = 10a + b$. From the rest of the statement of the problem, we have that

$$3(a+b)+(10a+b) = 10b+a$$
$$3a+3b+10a+b = 10b+a$$
$$12a = 6b$$
$$2a = b$$

If b is simply twice the value of a, then the possible numbers which satisfy the condition in the problem are: 12, 24, 36, and 48. We know that there cannot be any larger numbers since then it would not be a two-digit number.

3. Let A be the set of positive integers. If $S = \{x \in A : |2x-7| \leq 5\}$ and $T = \{x \in A : |3x-14| < 9\}$, how many elements are in $S \cup T$?

Solution: Beginning with set S, we have that

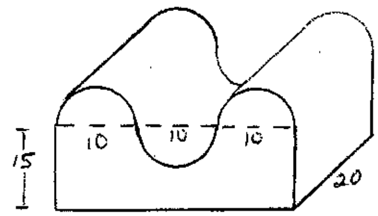
$$|2x-7| \leq 5$$
$$2x-7 \leq 5 \quad \text{or} \quad 2x-7 \geq -5$$
$$2x \leq 12 \quad \text{or} \quad 2x \geq 2$$
$$x \leq 6 \quad \text{or} \quad x \geq 1$$

This means that $S = \{1,2,3,4,5,6\}$. Similarly, we have for set T that

$$|3x-14| < 9$$
$$3x-14 < 9 \quad \text{or} \quad 3x-14 > -9$$
$$3x < 23 \quad \text{or} \quad 3x > 5$$
$$x < \tfrac{23}{3} \quad \text{or} \quad x > \tfrac{5}{3}$$

This means that $T = \{2,3,4,5,6,7\}$. Hence, $S \cup T = \{1,2,3,4,5,6,7\}$, which has 7 elements.

4. Find the surface area of the figure to the right and express your answer in terms of π. Note that the curve along the top edge forms three semicircles each with diameter 10.



Solution: We will compute the area of each side of the figure and then add them up to find the total surface area.

Sides: (2 rectangles) Area = $2(20 \times 15) = 600$

Bottom: (rectangle) Area = $20 \times 30 = 600$

Front and Back: (2 rectangles + 1 circle) Area = $2(30 \times 15) + \pi(5)^2 = 900 + 25\pi$

Top: (1.5 cylinder sides) Area = $1.5[2\pi(5) \times 20] = 300\pi$

In total, that gives $600 + 600 + 900 + 25\pi + 300\pi = 2100 + 325\pi$ square units.

5. Find all real and complex solutions to $x^3 - x^2 + 2 = 0$.

Solution: Observe that x =-1 solves the equation. Now we can apply synthetic division to get that

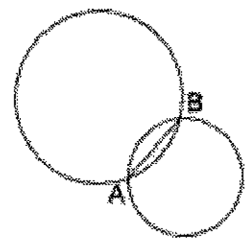
-1	1	-1	0	2
		-1	2	-2
	1	-2	2	0

This means that when we divide the original expression by $(x+1)$, we get a dividend of $x^2 - 2x + 2$. Applying the quadratic formula to this new polynomial, we get that

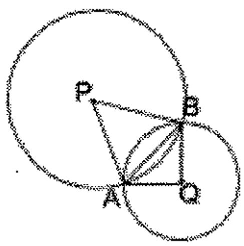
$$x = \frac{2 \pm \sqrt{4-4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

Therefore, the solution set is $x = -1,\ 1+i,\ 1-i$.

6. Two circles overlap as shown. The radius of the larger circle is 2 units and the radius of the smaller circle is $\sqrt{2}$ units. If the length of AB is 2 units, what is the area of the region where the two circles overlap?



Solution: Begin by drawing in the centers of each circle and then make triangles from the center of each circle to the points of intersection.



We know that the radius of the larger circle is 2, which means that the triangle drawn within it must be an equilateral triangle. Therefore, we know that angle APQ must be 60 degrees.

On the other hand, since the radius of the smaller circle is $\sqrt{2}$, we know that the triangle drawn within it must be a 45-45-90, where segment AB is the hypotenuse. This means angle AQB is 45 degrees.

Now, to find the area of the region of overlap, we simply need to sum up the areas of the 60 degree section of the large circle and the 45 degree section of the small circle and then subtract the areas of the two triangles APB and AQB.

<u>60 Degree Section</u>: Area = $\frac{1}{6}\pi(2)^2 = \frac{2}{3}\pi$
<u>45 Degree Section</u>: Area = $\frac{1}{4}\pi(\sqrt{2})^2 = \frac{1}{2}\pi$
<u>Triangle APB</u>: An equilateral triangle with side 2 has area $\frac{\sqrt{3}}{4}(2)^2 = \sqrt{3}$. (To see a derivation of this formula, see the solution for Round 4 Question 3).
<u>Triangle AQB</u>: A 45-45-90 triangle with legs of length $\sqrt{2}$ has area $\frac{1}{2}\left(\sqrt{2}\right)^2 = 1$.
<u>TOTAL</u>: the desired area is $\left(\frac{2}{3}+\frac{1}{2}\right)\pi - \sqrt{3} - 1 = \frac{7}{6}\pi - \sqrt{3} - 1$.

7. Factor completely: $A^2 + B^2 + 3A + 3B + 2AB - 4$

Solution: Begin by grouping terms

$$A^2 + B^2 + 3A + 3B + 2AB - 4$$
$$(A^2 + 2AB + B^2) + 3(A + B) - 4$$
$$(A + B)^2 + 3(A + B) - 4$$
$$(A + B + 4)(A + B - 1).$$

8. Laura can build a standard brick wall in exactly 9 hours by herself. Crystal can build the same size wall in 10 hours by herself. If they work together, their combined rate of labor decreases by 10 bricks laid per hour and they finish the wall in 5 hours. How many bricks are contained in one standard brick wall?

Solution: Let x be the number of bricks in a standard brick wall. We have that Laura's rate of bricklaying is $\frac{x}{9}$ bricks per hour and that Crystal's rate is $\frac{x}{10}$ bricks per hour. We are told that when they work together, their combined rate decreases by 10 bricks per hour and that they finish the wall - that is, they lay x bricks - in 5 hours. This means that

$$\left(\tfrac{x}{9}+\tfrac{x}{10}-10\right)\cdot 5 = x$$
$$\tfrac{5x}{9}+\tfrac{5x}{10}-50 = x$$
$$50x+45x-4500 = 90x$$
$$5x = 4500 \rightarrow x = 900.$$

9. Joe needs to cut a board which is 300 cm long into three different size pieces. The ratio of the length of longest piece to the length of the shortest piece is 5 : 3 and the ratio of the length of the middle size piece to the length of the shortest piece is 3 : 2. Find the length of the shortest piece.

Solution: The ratio of large to small is 5 to 3, which we can also express as 10 to 6. The ratio of middle to small is 3 to 2, which we can also express as 9 to 6. Hence, the desired ratio of large to medium to small is precisely 10 to 9 to 6.

Since $10+9+6=25$, divide the length of the board into 25 equal sections: $300 \div 25 = 12$. This means that the length of the shortest board is $6 \times 12 = 72$ cm.